A Coalgebraic Interpretation of the Social Welfare Functions as Collective Choice Rules

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Abstract. A significant part of the short history of social computing is related with the more recent part of the history in welfare economics, about the notion and the theory of the so-called "social welfare function (SWF)". It is intended as a function ranking social states as less, more, or indifferently desirable, for every pair of them, with respect to individual welfare measures and/or preferences.

One of the main uses of SWF is aimed, indeed, at representing coherent patterns (effectively, structures) of collective and social choices/preferences as to alternative social states.

The essential limitation of SWF's, also when considered as a subclass of the wider domain of the so-called "collective choice rules (CCR)", is that they are defined on *finite sets*, in the framework of an approach to the study of social and economic systems *stable at equilibrium*, because originally inspired from Samuelson's pioneering based on Gibbs' statistical thermodynamics of gases. We propose in this paper an alternative modelling based on the principle of dual equivalence algebra-coalgebra. This approach was born during the last twenty years from the initially independent, but now convergent research programs, on one side, for the mathematical modelling of condensed matter thermodynamic systems, stable in far from equilibrium conditions, in the framework of quantum field theory. On the other side, in theoretical computer science, for dynamic computation on infinite data stream, and for testing program security in functional programming. In this paper, we present some previous results for applying fruitfully such an approach also to social computing of CCR's, in the very "liquid" and complex actual social situation, where the necessity of fast and efficient computational models on infinite data streams is ever growing.

Keywords: social computing, social welfare functions, collective choice rules, semantic information, quantum field theory, quantum mechanics, quantum computations, coalgebraic modal logic, local truth, category theory, concurrent computations.

1 SOCIAL WELFARE FUNCTIONS AS COLLECTIVE CHOICE FUNCTIONS

1.1 Two main types of social welfare functions (SWF)

Generally, in economic literature, there are two main types of SWF's, as far as they are defined respectively, either on

1. Some support (domain-codomain) of real valued economical magnitudes (cardinal numbers) for one only social group, or
2. On orders, i.e., domains-codomains of rankings (ordered sets) of preferences, and of rankings (ordered sets) of social states.

Now, when we are speaking about "orders" effectively we are considering a SWF like a particular type of collective (social) choice function (CCF), relating ordered sets of individual preference/utilities, with ordered sets of individual social/economical states.

This means that, mathematically, we are moving from real valued (cardinal) functions, to set theoretic logic functions – effectively we are moving to set theoretic semantics (order theory) applied to social entities, i.e., to "social computing". The initial representatives of these two types of SWF theories generally quoted in literature are, respectively, "Bergson-Samuelson SWF" and "Arrow SWF".

1.2 Bergson-Samuelson SWF

Abram Bergson first introduced in economics the SWF notion as real-valued differentiable functions, aimed at formally representing "the conditions of maximum economic welfare" for the society as a whole. Bergson's SWF then includes as function arguments several quantities of different commodities produced and consumed, and of resources, labour included [1].

The fundamental contribution of the 1970 Nobel Prize in Economics Paul Samuelson – founder of the prestigious "MIT School of Economics" counting among its members an impressive lists of Nobel Prizes – is synthesized in what is commonly called the "Bergson-Samuelson SWF". It aims at representing (in the maximization calculus) all real-valued economic measures of any belief system – "that of a benevolent despot, or a complete egotist, or 'all men of good will', a misanthrope, the state, race, or group mind, God, etc." – required to rank consistently different feasible social configurations in an ethical sense as "better than", "worse than", or "indifferent to" each other [2].

What is essential for our aims is, anyway, that Samuelson's modelling of the equilibrium stability for economic systems explicitly depends on Willard Gibb's statistical mechanics interpretation of thermodynamic systems, as he explained in the first two chapters of his masterpiece *Foundations of economic analysis*, of which eighth chapter is dedicated to welfare economics and then to his SWF interpretation in such a framework [2].

Such a presupposition is generally shared by all models of equilibrium stability for economic systems, Kenneth J. Arrow's modelling included, even though in his approach this is explicitly related with Adam Smith's classical vision of an economic system, as guided by an intrinsic principle of fundamental balance between goods and services produced and consumed. Such a belief in the "Adam Smith invisible hand" is difficult to suppose for granted so easily today, because in a complex global market like ours where the "real time" information exchanges among "virtual" and "real" economic and social actors make hard (effectively, impossible) to suppose the usage of classical notion and measures of probability, necessary for studying economic systems as stable at equilibrium (see below,)[2].

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[2] In fact, Arrow was first to emphasize the role of the so-called "information asymmetry" sellers-buyers in the market dynamics [59], and

1.3 Arrow SWF

The second type of SWF is, indeed, related with the work of Kenneth J. Arrow, awarded with the Nobel Prize in economics on 1972. Since the first version of his theorem (1948), indeed, he transformed effectively Bergson SWF into a CCF, as the same title of his fundamental book *Social choice and individual value* exemplifies. That is, whereas the Bergson-Samuelson SWF rules the mapping from any set of individual orderings of preferences/utilities into one only set of ordering of social states, Arrow SWF rules the mapping from any set of individual orderings of preferences/utilities into a set of social states, among many alternative ones [3]. As Arrow himself emphasizes the interpretation of SWF as a CCF is effectively a restriction over SWF, because it requires that for any individual ordering, for some sufficiently wide but finite range of them, the SWF "give rise to a true social ordering" among finitely many.

Arrow's CCF effectively poses other conditions to SWF giving rise to the famous "Arrow's impossibility theorem". Following the later version of Arrow's theorem [3] they can be synthesized as follows:

- Let A be a set of alternatives of social states, N a number of individuals or preferences i, and $L(A)$ the set of all linear orderings of A. The SWF, intended as an individual preference aggregation rule, is a function F: $L(A)^N \rightarrow L(A)$ which aggregates individual preferences R_i on N into a single order on A. The N-tuple $(R_1, ..., R_N)$ of R_i is called a "(individual) preference ordering". The theorem states that for N at least of 2 individuals and for A at least of three alternatives the follow four conditions are incompatible – particularly the fourth as to the others:

1. *U*: Unrestricted domain*. The domain of the function (rule) F must include all logically possible individual orderings for a finite set of them;
2. *P*: Pareto efficiency* or "unanimity". All individual orderings have the same possibility of determining the social state ordering, i.e., if alternative x is ranked strictly higher than y for all orderings $R_1, ..., R_N$, then also for $F(R_1, ..., R_N)$;
3. *I*: Independence of irrelevant alternatives*. F depends strictly on the pairwise relations associating subsets of A and N, i.e., if for two different individual preference orderings R, S the alternatives x and y have the same order in R and R', then also in $F(R_1,..., R_N)$ and $F(S_1,..., S_N)$.
4. *D*: Non-dictatorship*. There is no individual $i \in \{1,\ldots N\}$ such that for $\forall (R_1,..., R_N) \in L(A)^N$ x ranked strictly higher than y implies that x is ranked strictly higher than y, for all x,y.

Now the "Arrow impossibility theorem" states that "there is no SWF satisfying simultaneously conditions U^*, P^*, I^* and D^*". Roughly speaking, given the condition P, for any individual set ordering there must be one *pivotal* individual that orders the alternatives. Now because of the transitivity and asymmetry of the *strict* individual orderings supposed in Arrow's construction, it is possible to demonstrate that this pivotal individual is *unique* for all individual orderings, so violating the condition D^*. The theorem had a tremendous impact overall in political sciences because, if taken for granted, it would demonstrate the irrationality of vote systems in democracy, as far as based on the principles of "majority" and of "representativeness". Anyway, two are the main ways for avoiding the outcome of Arrow's theorem:

1. Working on *infinite* sets of individuals and not on finite ones like in Arrow's SWF. In such a case, however, the aggregation rules are of limited interest because they are based on *ultrafilters* that are non-constructive mathematical objects, so as to suppose practically as many "invisible dictators" [4]. And, in fact, such rules violate also Turing computability [5], so to result practically useless in *social computing* applied to economics and to social sciences.
2. Working on finite sets *pre-orders* instead of strict orderings only, like in Arrow's SWF, so to define another type of *rational* CCF, the *social decision function* (SDF). For SDF's Arrow's impossibility theorem does not generally hold, even though it holds for SWF's as special cases of SDF's, as we see below. This approach was firstly proposed by another Nobel Prize in Economics (1998), Amartya Kumar Sen, who gave another essential contribution of clarification to our discussion [6]. We can anticipate, however, that also in his approach the "challenge of infinity" and hence of *effective computability* on infinite sets, e.g., data streams, represents itself as an *unsolvable computational problem*. At least till we are working in *standard* set theory and not in *non-standard* ones, like in the theory of *non-well founded* sets, as we see below.

1.4 Amartya Sen logic of preferences and SWF as SDF

Formally, his contribution to the discussion consists in a rigorous clarification of what "preferences/evaluations among alternatives" (e.g., "better than", "as good as", "at least good as", etc.) means in the framework of the logic (algebra) of relations, for solving ambiguities and inconsistencies that often appear in SWF theories interpreted as CCF. We illustrate Sen's contribution essentially following his outstanding book *Collective choice and social welfare* [6], in which he outlines a mathematical logic of preference/utility in economic and social sciences.

Whichever preference/evaluation binary relation R over a set S consists in specifying a subset R of the Cartesian Product $S \times S$, defined as the set of all ordered pairs (x, y) such that $\langle (x, y) \in S \rangle$.

Following the standard order theory in set theoretic logic, Sen defines rigorously the following relations R:

1. *Reflexivity:* $\forall x \in S$: xRx
2. *Completeness:* $\forall x,y \in S$: $(x \neq y) \rightarrow (xRy \vee yRx)$
3. *Transitivity:* $\forall x,y,z \in S$: $(xRy \wedge yRz) \rightarrow xRz$
4. *Anti-symmetry:* $\forall x,y \in S$: $(xRy \wedge yRx) \rightarrow x = y$
5. *Asymmetry:* $\forall x,y \in$ S: $xRy \rightarrow \neg (yRx)$
6. *Symmetry:* $\forall x,y \in$ S: $xRy \rightarrow yRx$

For logical and computational aims, it is worth to notice that asymmetry implies anti-symmetry, but not vice versa, so that the implication connective is logically correct only in the case of anti-symmetry. Now, Sen can define furtherly and newly in a standard way different types of "ordering relations", according to the different 1-5 relations they satisfy:

7. *Pre-ordering:* 1, and 3;
8. *Complete pre-ordering:* 1, 2, and 3;

the consequent "principal-agent problem" that is perfectly coherent with the outcome of his "impossibility theorem". Effectively, information asymmetry is fundamental in any communication process according to Shannon's statistical notion and measure of information, and not casually obtained the Nobel Prize in 2001 to George Akerlof, Michael Spence, and Joseph E. Stiglitz for their "analyses of markets with asymmetric information". However, it is highly significant for our aims that the impact of the growing role of AI agents in global markets has been recently studied as reducing information asymmetry, and then increasing market efficiency, because reducing the overall amount of exchanges related with information asymmetry distortions [58]. At the same time, such a situation makes increasingly difficult, if not despairing, the usage of classical statistical techniques of market analysis and previsions, as far as based on the "stability at equilibrium (or near-equilibrium)" principle.

9. *Partial ordering:* 1, 3, and 4;
10. *Complete ordering:* 1, 2, 3, and 4;
11. *Strict ordering:* 2, 3, and 5.

Sen defines thus the ordering relations, 7, 8, and 9 as "quasi-orderings" that is the key-notion of his logic of preferences, evidently based on the notion of "pre-orders" [6]. Then, Sen notices, in the case of Arrow's SWF, the "ordering" must effectively satisfy 1, 2, and 3, but it is, *per se,* irrespective of 4 and then of 5 [6]. In such a way, we can avoid, under some conditions explicated by Sen in the rest of the book that we only summarize here, the *uniqueness* of the "pivotal individual" for all social orderings, as depending on some rational CCF on social states he defined as "social decision function" (SDF). This opens the possibility of satisfying the condition *P* and *D* simultaneously of Arrow's theorem as characterizing any CCF, and then a SDF too, even though at the price of a social state organization much more "fluid", than in Arrow's vision.

In other terms, the core of Sen's approach consists in demonstrating formally that we can obtain suitable CCF's such as SDF's, satisfying all Arrow's conditions, without supposing *strict orderings*, but only "quasi-orderings".

Therefore, for the mathematical logic of CCF, according to Sen, it is important to distinguish three types of "preferences", characterizing "quasi-orderings", starting from the key-notion of "weak preference" *R* ("it is at least ___ as", e.g., "it is at least good as", or with a quantitative meaning, "it is at least great than")[3], which is the relation characterizing all quasi-orderings, and with respect to which the other two key-notions of "strict preference" *P* and "indifference" *I* of his logic derive. Indeed, given that the binary relation *R* of "weak preference" satisfies the relations 1, 2, and 3, but not 4, 5, and 6, Sen defines *P* and *I* as to *R* as follows:

12. *P: Strict preference*. $xPy \leftrightarrow (xRy \wedge \neg (yRx))$;
13. *I: Indifference*. $xIy \leftrightarrow (xRy \wedge (yRx))$.

Where "indifference" recalls here the notion of "anti-symmetry" for a generic relation *R* (see def. 4 above), i.e., of equality as to *R* of its arguments, evidently staying here as "indifference" of its arguments as to a weak preference *R*.

Other definitions significantly derived by Sen in its logic of preferences for a CCF are the followings:

14. *Maximal element. The elements of a set that are not dominated by any others in the set can be called the maximal elements of the set as to a given relation R. I.e.,*
    An element x in a set S is a maximal element of S as to a binary R, iff:
    $$\neg (\exists y\ (y \in S \wedge yPx)).$$
    The set of maximal elements in S is called its maximal set and denoted as M (S, R).
15. *Best element. An element x can be called a "best" ("greatest") element of S if it is "at least good (great) as" every other element in S, and as to the relevant preference relation R for such a set. I.e.,*
    An element x in S is a best element of S as to a binary relation R iff:
    $$\forall y: (y: (y \in S \rightarrow xRy)$$
    The set of best elements in S is called its choice set, and is denoted as C (S,R).

Of course, any best element is also a maximal element, but not vice versa, so that $C(S,R) \subset M(S, R)$. From this, the notion of "choice function" derives, as defined on a choice set and on all its subsets:

16. *Choice function. A choice function C (S, R) defined over X is a function relation such that the choice set C (S, R) is nonempty for every nonempty subset S of X.*

What is important to emphasize is that while *completeness* and *reflexivity* are two essential conditions to be satisfied for granting a choice function because granting that no non-empty set will exist in *X,* the same does not hold for *transitivity,* so to open the possibility of weaker transitivity sufficient conditions for a *C* (*S, R*) [6] such as the following two:

17. *Quasi-transitivity. If for all* $x,y,z \in X$ $xPy \wedge yPz \rightarrow xPz$, *then R is quasi-transitive.*

Where we recall that *R* is a weak-preference relation, of which *P* is a special case.

18. *Acyclicity. R is acyclical over X iff the following holds:*
    $$\forall x_1,\ldots,x_j \in X\left(\left(x_1Px_2 \wedge x_2Px_3 \wedge \ldots \wedge x_{j-1}Px_j\right) \rightarrow x_1Rx_j\right)$$

From this the following lemma *A** holds:

19. *Lemma A. If R is reflexive and complete, then a necessary and sufficient condition for C (S,R) to be defined over a finite X is that R be acyclical over X.*

Where it is to be noticed that quasi-transitivity implies acyclicity, where the converse does not follow. At this point, we can pass to define the essential notion of *collective choice function.* For this aim, it is important to introduce *two properties* generally characterizing any generic (not necessarily generated by a binary relation *R*) *rational choice function C* (*S*), defined by Sen as following:

20. *Property α*:
    $$x \in S_1 \subset S_2 \rightarrow \left(x \in C(S_2) \rightarrow x \in C(S_1)\right), \text{ for all } x$$
    that is another way for stating the condition of "independence of irrelevant alternatives".
21. *Property β*:
    $$\left(x, y \in C(S_1) \wedge S_1 \subset S_2\right) \leftrightarrow \left(x \in C(S_2) \leftrightarrow x \in C(S_2)\right) \text{ for all } x, y$$

With respect to a *choice function C* (*S,R*), it must always satisfy property *α*, but not necessarily property *β*. Indeed, there is a close relation between a choice function satisfying the property *β* and a particular notion of weak transitivity defined by Sen *PI*:

22. *PI-transitivity. A relation R is PI-transitive over X iff for all x,y,z in X,* $xPy \wedge yIz \rightarrow xPz$

From this, the following Lemmas *B*-D** hold for some *C* (*S,R*).

23. *Lemma B. A choice function C* (*S,R*) *generated by a binary relation R satisfies property β iff R is PI-transitive.*

Avoiding some steps of Sen's demonstration that can be found in [6], we arrive at the fundamental Lemmas:

24. *Lemma C. If a binary relation R generates a choice function, then PI-transitivity implies that R is an ordering.*

This holds essentially because *PI-transitivity* implies transitivity. Therefore:

25. *Lemma D*: *A choice function C* (*S,R*) *derived from a binary relation R satisfies property β iff R is an ordering.*

If a choice function satisfies the Pareto condition *P** of unanimity as an expression of *social consensus,* then we can define a *collective choice rule* (CCR) as follows [6]:

26. *A CCR is a functional relation f such that for any set of n individual orderings* $R_1, \ldots, R_n$ *(one ordering for each individual), one and only one social preference relation R is determined,* $R = f(R_1, \ldots, R_n)$.

Where it is to be emphasized that the resulting *R is not* a *social ordering.* Therefore,

27. *A CCR is decisive iff its range is restricted to complete preference relation R.*

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[3] To use the same symbol *R* for denoting a *generic relation* deriving from general order theory as effectively used till now in the definitions 1-11, and from this point on as denoting a *weak preference relation,* say *R**, can determine some ambiguity in the symbolism. Nevertheless, we follow Sen in this usage of the symbol *R* without any substantial problem with the only advertising that in Sen theory on CCR, that is, from this point on it has always the sense of *R**.

It is essential to emphasize that a CCR is a SWF iff all social preferences are orderings, satisfying conditions 1, 2 and 3 above. Transitivity (i.e., (3)), indeed is essentially condition on "triples", given that acyclicity could hold over *all* triples, and yet may violate acyclicity over the entire set [6].

However, Sen notices, given that "impossibility theorem" holds only for SWF, if the aim, as Arrow himself stated, is only to ensure that "from any environment, there will be a chosen alternative" [3], a CCR able to grant only social preferences sufficient for the existence of choice functions would be sufficient. Such a CCR is defined by Sen as *social decision function* (SDF) [6]:

28. *A SDF is a CCR f, the range of which is restricted to those preference relations R, each of which generates a choice function C* (*S,R*) *over the whole set of alternative X.*

It is evident that all SWF are SDF, but the converse is not true. Indeed, for SDF's no "impossibility result" holds. Indeed, a SDF satisfies simultaneously all four conditions *U*, P*, I*, D** defined above of Arrow's "possibility theorem" [6]. Nevertheless,

29. *For an infinite set X there is no SDF satisfying conditions U* and P*.*

In other terms, both for SDF and SWF the condition of finiteness of *X* holds, because in the case of an infinite chain of subsets *S* of *X* no best element can exist. On the other hand, if both SDF and SWF satisfy the property α (20) of a rational social choice, nevertheless any SDF generating choice functions satisfying also the property β (21) is a SWF, because, for Lemma *D* (25), *R* in such a case becomes an *ordering relation*. In such a case, for a SDF that is also a SWF, the "impossibility theorem" holds [6].

Finally, other two interesting consequences derive from the finiteness of CCR's, is the possibility of representing in it also *judgements about values,* shared by a homogeneous social group because characterized by strong *interpersonal exchanges of information* [6]. These topics were the main object of a more recent essay of Sen, with the significant title: "The Informational Basis of Social Choice". However, when we study social and economic systems, taking into account also the *motivational social forces* acting within them, and vehiculated in *real time* by social media, internet before all, the *challenge of infinity* represents itself because making untenable that either a SDF or a SWF can range over a *finite set* of social state alternatives, chosen by *individual* preference orderings. Such a "liquid" situation of our society and of our economy requires a modelling more based on *fluid thermodynamics,* than on *gas thermodynamics*.

All this requires a *paradigm shift* in representing social and/or economical systems and in modelling their always changing stability conditions, from Gibbs' *statistical approach* of gas thermodynamic systems stable at equilibrium (see before §1.2), to a *dynamic approach* of condensed matter systems, stable in *far from equilibrium conditions*. In a word, the shift from the thermodynamic paradigm of classical statistical mechanics, to the thermodynamic paradigm of the *thermal field theory.*

1.5 The challenge of infinity in social computing and the paradigm shift in quantum computing

Recently in an essay published on a collective book dedicated at a balance about Arrow's impossibility theorem, almost seventy years after its first publication, and edited by E. Maskin and A. Sen, K. Arrows himself focused, in his reply to other discussants, which is the main challenge of social computing today:

> When you are dealing with infinite dimensional elements, can you really compute the results? Some things are simply quite extremely difficult to compute. They are not constructible in the sense that there is no finite process that will enable an individual to carry out the calculation. This applies to a lot of problems, not just those that are social in nature, such as climate change, but also to individual as well as social choice problems [7].

It is paradoxical, but significant that just the condensed matter physics and the related Quantum Field Theory (QFT), interpreted as a thermal field theory, and its mathematical formalism developed during the last twenty years by theoretical physicists is, able to give, at least in principle an answer to the issue of Arrow. Simultaneously, but independently, theoretical computer scientists developed a model of concurrent computations, based on the same principle of the duality algebra-coalgebra of QFT. We dedicate the next two Sections to illustrating such an approach deeply different from Quantum Mechanics (QM) paradigm in quantum computing.

In the next Section we sketch some fundamental concepts of QFT, while in the fourth one we discuss QFT formalism in the framework of theoretical computer science, for synthesizing in the conclusions the main results that we could obtain from such an approach when applied to social computing issues.

2 FROM QM TO QFT IN FUNDAMENTAL PHYSICS

2.1 From mechanical vacuum to quantum vacuum

In the official conference press for announcing to the world that the 2015 Nobel Prize in Physics was awarded to the physicists T. Kajita and A. B. McDonald for their observational discovery of the neutrino mass, the Academy stated that "the new observations had clearly showed that the Standard Model cannot be the complete theory of the fundamental constituents of the universe" [8]. From the theoretical standpoint, the best candidate to such a shift of paradigm in fundamental physics is QFT. Till now it was conceived as an extension of QM, i.e., the so-called "second quantization" of P. Dirac and R. Feynman. In this interpretation, in the study of the fundamental electromagnetic interactions of the "Quantum Electrodynamics" (QED) and of the color charge interactions of the strong force of "Quantum Cromodynamics" (QCD), it continues to work according to the mechanistic scheme of the Newtonian Mechanics (particles isolated from forces in the mechanical vacuum) that, from Laplace on, in the study of many body dynamic systems, uses systematically the so-called "perturbation methods".

They study the particle behaviour using systematically the so-called "asymptotic condition", i.e., they represent the system by separating the objects at infinite spatio-temporal distances, so to isolate the particles from interactions "cutting-off" them, and re-create artificially the condition of particle isolation of the Newtonian Mechanics. The supposition is that such a modelling does not falsify the observed phenomena. The consequence of such an approach is the absolute distinction between particles and interaction force fields, constituting the core also of the Standard Model.

In other terms, in the Standard Model the ontological distinction particle-force is introduced by interpreting the distinction, in itself only statistical, between fermions and bosons like the difference between particles constituting the "building bricks" of ordinary matter (quarks and leptons (electrons, neutrinos, etc.)), divided into three families or generations (the first three columns of the figure below), and quanta of the three fundamental quantum forces electromagnetic and nuclear strong and weak (photons, gluons and bosons Z and W, respectively), with the addition of the Higgs field with the relative boson. Anyway, the empirical evidence of neutrino mass, united to the growing disaffection of physicists for the perturbative methods in quantum physics because it is impossible to considerate a quantum system as isolated from the QV fluctuations in which it is immersed from within, are the deep reasons for the growing interest to the alternative paradigm of QFT, "QFT can be recognized as an *intrinsically thermal* quantum theory" [9].

According to this alternative picture of QFT, every particle, both fermionic or bosonic, is considered as the quantum of the relative force field. Roughly speaking, there exist material force fields (fermionic) and interaction force fields (bosonic). In this framework, the suggestion as to the neutrino oscillations is that they consist in as many phase transitions of the same neutrino field.

On the other hand, the association of whichever mole of matter to a force field, and therefore the existence of the QV as the totality of the quantum force fields is an immediate consequence of the Third Principle of Thermodynamics. It affirms that for whichever physical system it is impossible to reach the absolute 0°K. This means that near the absolute 0°K, there is a mismatch between the variation of the body content of energy, and the supply of energy from the outside. We can avoid such a paradox, only by supposing that such a mysterious inner supplier of energy is the vacuum. Intuitively, the QV can be interpreted as a sort of universal "energy reservoir" of all energy forms in the universe(s) (the temperature of the QV is indeed >0°K, even though it is bounded energy, for the lack of any "ordering"), as something including and connecting dynamically everything. In this framework, any physical system at whichever degree of complexity is immersed "from within" into the QV.

From the mathematical standpoint, the main difference between QM and QFT is that in the latter the fundamental Stone-Von Neumann theorem [10] does not hold. This theorem states that, for system with a *finite* number of degrees of freedom, which is always the case in QM, the representations of the canonical commutation relations[4] are all *unitarily equivalent to each other*.

On the contrary, in QFT systems, the number of the degrees of freedom is not finite, so that infinitely many unitarily inequivalent representations of the canonical commutation (bosons) and anti-commutation (fermions) relations exist. Indeed, through the principle of the *Spontaneous Symmetry Breakdown* (SSB) in the vacuum ground state, infinitely (not denumerable) many, quantum vacuum conditions, compatible with the QV ground state, there exist. Moreover, this holds not only in the relativistic (microscopic) domain, but also it applies to non-relativistic many-body systems in condensed matter physics, i.e., in the macroscopic domain, and even on the cosmological scale [9].

Indeed, starting from the discovery, during the 60's of the last century, of the dynamically generated long-range correlations mediated by the *Nambu-Goldstone bosons* (NGB) [11,12], and hence for their role in the local gauge theory by the Higgs field, the discovery of these collective modes changed deeply the fundamental physics.

Of course, because of the intrinsic character of the thermal bath, the whole QFT system can recover the classical Hamiltonian character, because of the necessity of anyway satisfying the energy balance condition of each QFT (sub-)system with its thermal bath ($\Delta E = 0$), mathematically formalized by the "algebra doubling", between a q-deformed Hopf algebra and its "dual" q-deformed Hopf co-algebra, where q is a thermal parameter [13].

Therefore, in QFT an uncertainty relation holds, similar to the one of Heisenberg, relating the uncertainty on the number of the field quanta to the one of the field phase, namely:

$$\Delta n \Delta \varphi \geq \varphi \left(\hbar \right)$$

Where n is the number of quanta of the force field, and φ is the field phase. If ($\Delta n = 0$), φ is undefined so that it makes sense to neglect the waveform aspect in favour of the individual, particle-like behaviour. On the contrary if ($\Delta\varphi = 0$), n is undefined because an extremely high number of quanta are oscillating together according to a well-defined phase, i.e., within a given phase coherence domain. In this way, it would be nonsensical to describe the phenomenon in terms of individual particle behaviour, since the collective modes of the force field prevail.

In QFT there is duality between *two dynamic entities*: the fundamental force field and the associated quantum particles that are simply the quanta of the associated field that is different for different types of particles. In such a way, the quantum entanglement does not imply any odd relationship between particles like in QM, but simply it is an expression of the unitary character of a force field. To sum up, according to such more coherent view, Schrödinger wave function of QM appears to be only a statistical coverage of a finest structure of the dynamic nature of reality.

2.2 Order and vacuum symmetry breakdowns

It is well-known that a domain of successful application of QFT is the study of the microphysics of condensed matter, that is in systems displaying at the macroscopic level a high degree of coherence related to an *order parameter.* The "order parameter", that is the macroscopic variable characterizing the new emerging level of matter organization, is related to the *matter density distribution*. In fact, in a crystal, the atoms (or the molecules) are "ordered" in well-defined positions, according to a *periodicity law* individuating the crystal lattice.

Other examples of such ordered systems in condensed matter realm are the magnets, the lasers, the super-conductors, etc. In all these systems the emerging properties related to the respective order parameters, are neither the properties of the elementary constituents, nor their "summation", but new properties depending on *the modes in which they are organized*, and hence on *the dynamics controlling their interactions.* In this way, at each new macroscopic structure, such a crystal, a magnet or a laser, corresponds a new "function", the "crystal function", the "magnet function", etc.

Moreover, all these emerging structures and functions are controlled by *dynamic parameters,* that, with an engineering terminology, we can define as *control parameters.* Changing one of them, the elements can be subject to different dynamics with different collective properties, and hence with different collective behaviours and functions. Generally, the temperature is the most important of them. For instance, crystals beyond a given critical temperature — that is different for the different materials — lose their crystal-like ordering, and the elements acquire as a whole the macroscopic structure-functions of an amorphous solid or, for higher temperatures, they lose any static structure, acquiring the behaviour-function of a gas.

[4] It is useful to recall here that the *canonical variables* (e.g., position and momentum) of a quantum particle do not commute among themselves, like in classical mechanics, because of Heisenberg's uncertainty principle. The fundamental discovery of D. Hilbert consists in demonstrating that each canonical variable of a quantum particle commutes with the Fourier transform of the other (such a relationship constitutes a canonical commutation relation), so to allow a geometrical representation of all the states of a quantum system in terms of a commuting variety, i.e., the orthonormal finite basis of a *per se* infinite dimensional "Hilbert space".

So, any process of *dynamic ordering,* and of *information gain,* is related with a process of *symmetry breakdown.* In the magnet case, the QV "broken symmetry" is the rotational symmetry of the magnetic dipole of the electrons, and the "magnetization" consists in the correlation among all (most) electrons, so that they all "choose", among all the directions, that one proper of the magnetization vector.

To sum up, whichever dynamic ordering among many objects implies an "order relation", i.e., a *correlation* among them. What, in QFT, at the *mesoscopic/macroscopic* level is denoted as *correlation waves* among molecular structures and their chemical interactions, at the *microscopic* level any correlation, and more generally any interaction, is mediated by *quantum correlation particles,* i.e., NGB's [14,11,12], with mass — even though always very small (if the symmetry is not perfect in finite spaces) —, or *without mass at all* (if symmetry is perfect, in the abstract infinite space). Less is the inertia (mass) of the correlation quantum, greater is the distance on which it can propagate, and hence the distance on which the correlation (and the ordering relation) constitutes itself.

However, an important *caveat* is necessary to do about the different role of the "Goldstone bosons" as quantum correlation particles, and the "bosons" of the different energy fields of quantum physics (QED and QCD). These latter are the so-called *gauge bosons*: the photons γ of the electromagnetic field; the gluons g of the strong field, the bosons $W^{\pm}$ and the boson Z of the electroweak field; and the scalar Higgs boson H^0 of the Higgs field, common to all the precedent interactions.

The gauge bosons are properly mediators of the *energy exchanges,* among the interacting elements they correlate, because they are effectively quanta of the energy field they mediate (e.g., the photon is the quantum of the electromagnetic field). Therefore, the energy quanta are bosons able to change the *energy state* of the system. For instance, in QED of atomic structures, they are able to change the fundamental state (minimum energy), into one of the excited states of the electronic "cloud" around the nucleus.

On the contrary, NGB correlating quanta are not mediators of the interactions among the elements of the system. They determine only the *modes of interaction* among them. Hence, any symmetry breakdown in the QFT of condensed matter of chemical and biological systems has one only gauge boson mediator of the underlying energy exchanges, the photon, since they all are electromagnetic phenomena. Therefore, the phenomena here concerned, from which the emergence of *macroscopic* coherent states derives, implies the generation, effectively the *condensation,* of correlation quanta with negligible mass, in principle null: the NGB, indeed. This is the basis of the fundamental "Goldstone theorem" [15,16]. Therefore, despite the correlation quanta are real particles, observable with the same techniques (diffusion, scattering, etc.), not only in QFT of condensed matter, but also in QED and in QCD like the other quantum particles, wherever we have to deal with broken symmetries [12], nevertheless they do not exist *outside* the system they are correlating. For instance, without a crystal structure (e.g., by heating a diamond over 3,545 °C), we have still the component atoms, but no longer phonons. Also and overall in this aspect, the correlation quanta differ from energy quanta, like photons. Because the gauge bosons are *energy* quanta, they cannot be "created and annihilated" without residuals.

Better, in any quantum process of particle "creation/annihilation" in quantum physics, what is conserved is the energy/matter, mediated by the energy quanta (gauge bosons), not their "form", mediated by the NGB correlation quanta. Also on this regard, a dual ontology (matter/form) is fundamental for avoiding confusions and misinterpretations in quantum physics.

Moreover, because the mass of the correlation quanta is in any case negligible (or even null), *their condensation does not imply a change of the energy state of the system.* This is the fundamental property for understanding how, not only the stability of a crystal structure, but also the relative stability of the living matter structures/functions, at different levels of its self-organization (cytoskeleton, cell, tissue, organ…), can depend on such basic *dynamic* principles. In fact, all this means that, if the symmetric state is a fundamental state (a minimum of the energy function corresponding to a *quantum vacuum* in QFT of dissipative systems), also the ordered state, after the symmetry breakdown and the instauration of the ordered state, remains *a state of minimum energy*, so to be *stable* in time. In kinematics terms, it is a *stable attractor* of the dynamics.

2.3 The Doubling of Degrees of Freedom (DDF) in QFT and in neuroscience

We said that the relevant quantum variables in biological systems are the *electrical dipole vibrational modes* in the water molecules, constituting the oscillatory "dynamic matrix" in which also neurons, glia cells, and the other mesoscopic units of the brain dynamics are dipped. The condensation of massless NGB (polarons) — corresponding, at the mesoscopic level, to the long-range correlation waves observed in brain dynamics — depends on the triggering action of the external stimulus for the symmetry breakdown of the quantum vacuum of the corresponding brain state. In such a case, the "memory state" corresponds to a coherent state for the basic quantum variables, whose mesoscopic order parameter displays itself as the amplitude and phase modulation of the carrier signal.

At this point emerges the principle of the *doubling of the degrees of freedom* (DDF) between a quantum system and its thermal bath as a general principle of all QFT systems, which, however, we illustrate here as to QFT brain dynamics, because closer to our computational aims. This principle emerges as a both physical and mathematical necessity of QFT modelling. Physical, because a dissipative system, even though in non-equilibrium, must anyway satisfy the *energy balance*. Mathematical, because the 0 energy balance requires a "doubling of the system degrees of freedom". The *doubled* degrees of freedom, say $\tilde{A}$ (the tilde quanta, where the non-tilde quanta A denote the brain degrees of freedom), thus represent the environment to which the brain state is coupled. The environment (state) is thus represented as the "time-reversed *double*" of the brain (state) on which it is impinging. The environment is hence "modeled on the brain", but according to the finite set of degrees of freedom *the environment itself elicited* in the brain.

What is relevant for our aims, is that to each set of degrees of freedom A and to its "entangled doubled" $\tilde{A}$ is relative a *unique number* $\mathcal{N}$, i.e. $\mathcal{N}_A, \mathcal{N}_{\tilde{A}}$ that in module, $|\mathcal{N}|$, *identifies univocally*, i.e., it *dynamically labels*, a given *phase coherence domain*, i.e., a quantum system state entangled with its thermal bath state, in our case, *a brain state matching its environment state.* This depends on the fact that generally, in the QFT mathematical formalism the number $\mathcal{N}$ is a numeric value expressing the NGB condensate value from which a phase coherence domain *directly depends*. In an appropriate *set theoretic interpretation,* because for each "phase coherence domain" *x,* effectively $|\mathcal{N}|$ *identifies univocally* such a domain, it corresponds to an "identity function Id_x" that, in a "finitary" coalgebraic logical calculus, corresponds to the *predicate satisfied by such a domain because identifying univocally it.* In other terms, Vitiello's reference to the predicate "magnet function" or "crystal function" we quoted at the beginning of sect. 2.2 are not metaphors, but are expressions of a fundamental formal tool – the "co-membership notion" – of the coalgebraic predicate calculus (see below sect. 3.2). Therefore, of the DDF applied to the quantum foundation of the cognitive neuroscience we have illustrated elsewhere its logical relevance, for an original solution of the reference problem (see [17,18]).

There exists a huge amount of experimental evidence in brain dynamics of such phenomena, collected by W. Freeman and his collaborators. This evidence found, during the last ten years, its proper mathematical modelling in the dissipative QFT approach of Vitiello and his collaborators, so to justify the publication during the last years of several joint papers on these topics (see, for a synthesis, [19,20]).

2.4 QFT systems and the notion of negentropy

Generally, the notion of information in biological systems is a synonym of the *negentropy* notion, according to E. Schrödinger's early use of such a term. Applied, however, to QFT foundations of dissipative structures in biological systems, the notion of negentropy is not only associated with the *free energy,* as Schrödinger himself suggested [21], but also with the notion of *organization,* as the use of this term by A. Szent-György first suggested [22]. The notion of negentropy it is thus related with the constitution of *coherent domains* at different space-time scales, as the application of QFT to the study of dissipative structures demonstrates, since the pioneering H. Frölich works [23,24].

On this regard, it is important to emphasize also the key-role of the notion of *stored energy* that such a multi-level spatial-temporal *organization* in coherent domains and sub-domains implies (i.e., the notion of quantum vacuum "foliation" in QFT), as distinct from the notion of *free energy* of classical thermodynamics [25]. Namely, as we know from the precedent discussion, the constitution of coherent domains allows chemical reactions to occur at *different time-scales*, with a consequent energy release, that so becomes immediately available exactly *where/when it is necessary*. For instance, resonant energy transfer among molecules occurs typically in 10^{-14} sec., whereas the molecular vibrations themselves die down, or thermalize, in a time between 10^{-9} and 10^{1}sec. Hence, it is a 100% highly efficient and highly specific process, being determined by the frequency of the vibration itself, given that resonating molecules can attract one another. Hence, the notion of "stored energy" is meaningful at every level of the complex spatial-temporal structure of a living body, from the single molecule to the whole organism.

This completes the classical thermodynamic picture of L. Szilard [26] and L. Brillouin [27] according to which the "Maxwell demon", for getting information so to compensate the entropic decay of the living body must consume free energy from the environment. This means an increasing of the global entropy according to the dictate of the Second Law. However, this has to be completed in QFT with the evidence coming from the Third Law discussed in this paper.

3 COALGEBRAIC SEMANTICS OF QUANTUM SYSTEMS

3.1 Category theory logic and coalgebraic semantics

To illustrate the fundamental principles of QFT computing, let us start from the fundamental observation that in QFT, because of the dynamic field reinterpretation of the particle-wave quantum duality introduced in §2.1, the probabilities of the quantum states follow a Wigner distribution, based on the notion and the measure of *quasi-probability* where regions integrated under given expectation values do not represent mutually exclusive states. This means that a computing agent, either natural or artificial in QFT, against the Quantum Turing Machine paradigm, *is able to change dynamically the representation space of its computations* for matching *dynamically* (automatically) the hidden degrees of freedom of the data set (thermal bath). This depends on the possibility of interpreting the QFT system computations within the framework of the Category Theory (CT) logic and its principle of duality between opposed categories, such as the algebra and coalgebra categories of the QFT.

This justifies in principle the interpretation of the *maximal entropy* in a QFT "doubled" system as a semantic measure of information, i.e., as a statistical measure of *maximal local truth* in the CT coalgebraic logic. In the QFT mathematical formalism this maximum of the entropy measure is formally obtained when the above illustrated DDF principle (far from equilibrium energy balance) between a system (algebra) and its thermal bath (coalgebra) is *dynamically* satisfied. This means that we are allowed to interpret the QFT *qubit* of such a natural computation as an "evaluation function" in the semantic sense. Indeed, in the QFT "composed Hilbert space" including also the thermal bath degrees of freedom, $\tilde{A}$, i.e. $\mathcal{H}_{A,\tilde{A}} = \mathcal{H}_A \otimes \mathcal{H}_{\tilde{A}}$, for calculating the static and dynamic entropy associated with the time evolution generated by the free energy, i.e., $|\phi(t)\rangle, |\psi(t)\rangle$, of the qubit mixed states $|\phi\rangle, |\psi\rangle$, one needs to double the states by introducing the tilde states $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$, relative to the thermal bath, i.e., $|0\rangle \to |0\rangle \otimes |\tilde{0}\rangle$, and $|1\rangle \to |1\rangle \otimes |\tilde{1}\rangle$. This means that such a QFT version of a qubit implements effectively the CNOT (controlled NOT) logical gate, which flips the state of the qubit, conditional on a *dynamic* control of an effective input matching [28,9].

What is highly significant for our aims is that in a way completely independent from quantum physicists – at least till the very last years – logicians and computer scientists developed in the context of CT logic *a coalgebraic approach to Boolean algebra semantics* that only recently started to be applied also to quantum computing. Let us start from some basic notions of the CT logic (for a survey, see [29]).

The starting point of such a logic as to set theory is that the fundamental objects of CT are not "elements" but "arrows", in the sense that also the set elements are always considered as domains-codomains of *arrows* or *morphisms* – in the case of sets, domains-codomains of *functions*.

In this sense, any object *A, B, C,* characterizing a category, can be substituted by the correspondent *reflexive morphism* $A \to A$ constituting *a relation identity* Id_A. Morover, for each triple of objects, *A,B,C,* there exists a *composition map* $A \xrightarrow{f} B \xrightarrow{g} C$, written as $g \circ f$ (or sometimes: f ; g), where *B* is the codomain of *f* and domain of *g*[5]. Therefore, a *category* is any structure in logic or mathematics with structure-preserving morphisms. E.g., in set theoretic semantics, all the models of a given formal system because sharing the same structure constitute a category. In this way, some fundamental mathematical and logical structures are as many categories: **Set** (sets and functions), **Grp** (groups and homomorphisms), **Top** (topological spaces and continuous functions), **Pos** (partially ordered sets and monotone functions), **Vect** (vector spaces defined on numerical fields and linear functions), etc.

[5] We recall that typical example of function composition is a recursive, iterated function: $x_{n+1} = f(x_n)$.

Another fundamental notion in CT is the notion of *functor, F,* that is, an operation mapping objects and arrows of a category **C** into another **D**, *F:* **C** → **D,** so to preserve compositions and identities. In this way, between the two categories there exists a *homomorphism up to isomorphism*. Generally, a functor *F* is *covariant,* that is, it preserves arrows directions and composition orders (e.g., in the QM attempt of interpreting thermodynamics within kinematics [30]), i.e.:

if $f : A \to B$, then $FA \to FB$;

if $f \circ g$, then $F(f \circ g) = Ff \circ Fg$; if id_A, then $Fid_A = id_{FA}$.

However, two categories can be equally homomorphic up to isomorphism if the functor *G* connecting them is *contravariant,* i.e., *reversing* all the arrows directions and the composition orders, i.e. *G*: **C** → $\mathbf{D^{op}}$:

if $f : A \to B$, then $GB \to GA$; if $f \circ g$, then $G(g \circ f) = Gg \circ Gf$; but if id_A, then $Gid_A = id_{GA}$.

Through the notion of contravariant functor, we can introduce the notion of *category duality.* Namely, given a category **C** and an *endofunctor E*: **C** → **C**, the contravariant application of *E* links a category to its opposite, i.e.: E^{op}: **C** → $\mathbf{C^{op}}$. In this way it is possible to demonstrate the *dual equivalence* between them, in symbols: $\mathbf{C} \rightleftharpoons \mathbf{C^{op}}$. In CT semantics, this means that given a statement α defined on **C** α is true *iff* the statement α^{op} defined on $\mathbf{C^{op}}$ is also true. In other terms, truth is invariant for such an exchange operation over the statements, that is, they are *dually equivalent.* In symbols: $\alpha \rightleftarrows \alpha^{op}$, as distinguished from the ordinary equivalence of the logical tautology: $\alpha \leftrightarrow \beta$, defined within the very same category. We can anticipate here that the physical basis of this notion is precisely the *energy balance* between a system and its *thermal bath*, as far as interpreted as the *duality* between an algebra and its coalgebra, given that it is standard in modern physics to model physical systems through algebraic (and now, more effectively, coalgebraic) structures.

A particular category, indeed, that is interesting for our aims is the category of Algebras, **Alg.** They constitute a category because any algebra $\mathcal{A}$, can be defined as *a structure defined on sets* characterized by an endofunctor projecting all the possible combinations (Cartesian *products*) of the subsets of the carrier set, on which the algebra is defined, onto the set itself, that is, $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. The other category interesting for us is the category of coalgebras **Coalg.** Generally, a coalgebra can be defined as a structure defined on sets, whose endofunctor projects from the carrier set onto the *coproducts* of this same set, i.e., $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Despite the appearances, an algebra and its coalgebra *are not dual.* This is the case, for instance, of a fundamental category of algebras in physics, that is, the *Hopf Algebras,* **Halg,** generally used in dynamic system theory both in classical and in quantum mechanics, as we know. Each *HAlg* is essentially a *bi-algebra* because including two types of operations on/to the carrier set, where – because used to represent energetically closed systems – products (algebra) and coproducts (coalgebra) can be defined on the same basis, and therefore *commute* among themselves. That is, there exists a complete *symmetry* between a *HAlg* and its *HCoalg* so that they are *equivalent* and not *dually equivalent.* In this sense any Hopf algebra is said to be *self-dual,* that is, isomorphic with itself. To make dually equivalent a Hopf algebra with its coalgebra, as we know from QFT, we have to introduce a *q*-deformation, where *q* is a thermal parameter.

More generally, indeed, it is possible to define a dual equivalence between two categories of algebras and coalgebras by a contravariant application of the same functor. This is particularly significant whereas it is meaningless that both are defined on the same basis, and therefore products and coproducts do not commute among themselves. Two are the examples that we might give of this notion, the former in mathematics and computability theory concerning Boolean algebras, the second in computational physics concerning QFT.

3.2 Coalgebraic semantics of a Boolean logic for a contravariant functor

The first example concerning Boolean algebras depends essentially of the fundamental representation theorem for Boolean algebras demonstrated in 1936 by the American Mathematician M. Stone, five years after having demonstrated with John Von Neumann the fundamental theorem of QM we quoted in sect. 2. Indeed, the Stone theorem, associates each Boolean algebra *B* to its Stone space *S*(*B*) [31]. Therefore, the simplest version of the Stone representation theorem states that every Boolean algebra *B* is *isomorphic* to the algebra of partially *ordered by inclusion* closed-open (clopen) subsets of its Stone space *S*(*B*), effectively an *ultrafilter*[6] of the power set of a *given set (interval) of real numbers* defined on *S*(*B*). The ultrafilters constituting an insuperable computational problem for both SDF and SWF as defined on infinite sets but necessary for the actual states of social and economic modelling (see above §1.5).

Because, each homomorphism between a Boolean algebra *A* and a Boolean algebra *B* corresponds to a continuous function from *S*(*B*) to *S*(*A*), we can state that each endofunctor Ω in the category of the Stone spaces, **Stone** (where the objects are Stone spaces and the arrows are continuous functions)**,** *induces* a contravariant functor Ω* in the category of the Boolean algebras, **BAlg** (where the objects are Boolean algebras and the arrows are recursive functions). In CT terms, the theorem states the *dual equivalence* between them, i.e., **Stone(Ω)⇌BAlg(Ω*).**

It is difficult to exaggerate the fundamental importance of the Stone theorem that, according to the computer scientists, inaugurated the "Stone era" in computer science. Particularly, this theorem demonstrated definitively that Boolean logic semantics requires only *a first-order semantics* because it requires only *partially ordered* sets and not *totally ordered sets*. This result is particularly relevant for the foundations of computability theory. Indeed, the demonstration of the fundamental Lövenheim-Skolem theorem (1921) blocked the research program of E. Schröder of the so-called "algebra of logic" in the foundations of mathematics and of calculus [32], because it demonstrated that algebraic sets are not able to deal with *non-denumerable sets*, e.g., with the *totality* of real numbers. For this reason, and the subsequent fundamental demonstrations of Tarski's theory of truth as correspondence (1929) [33], and of Gödel's incompleteness theorems (1931) [34], the set-theoretic semantics migrated to higher-order logic, so to grant the *total ordering* of sets, by some foundation axiom, e.g., the *axiom of regularity* in ZF. In this way, no *infinite chain of inclusions* among sets is allowed in *standard* set theory, so to separate the semantic "set ordering" from the complete "set enumerability"[7].

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[6] We recall here that by an "ultrafilter" we intend the maximal partially ordered set defined on the power-set of a given set ordered by inclusion, and excluding the empty set.

[7] Two corollaries of the Lövenhiem-Skolem theorem, demonstrated by Skolem himself in 1925 are significant for our aims, i.e., 1) that only *complete* theories are *categorical*, and 2) that the *cardinality* of an algebraic set depends intrinsically by the algebra defined on it. Think, for instance at the principle of *induction by recursion* for Boolean algebras, allowing a Boolean algebra to *construct* the sets on which its semantics is justified, blocking however Boolean computability

Therefore, the further step for making computationally effective the Stone theorem for a Boolean first-order semantics, avoiding the limits of the Turing-like computation scheme strictly dependent on Gödel and Tarski theorems, is the definition of *non-standard* set theories *without foundation axioms*. In this way, we allow infinite chains of set inclusions, according to the original intuition of the Italian mathematician E. De Giorgi [35,36]. The most effective among the non-standard set theory is Aczel's set theory of *non-wellfounded (NWF)sets* based on the *anti-foundation axiom* (AFA) [37]. AFA, indeed, allowing set *self-inclusions* and therefore infinite chains of set inclusions, makes also possible to define the powerful notion of set *co-induction* by *co-recursion, dual* to the algebraic notion of *induction* by *recursion,* both as formal methods of set definition and proof [38,39,36] (See below Appendix 5.1). This immediately suggests us the coalgebraic solution of Arrow's issue in §1.5 about the impossibility of a constructive (i.e., algebraic) approach to infinite data streams in social computing. Using two concurrent coalgebraic (co-inductive) and algebraic (inductive) computations implemented in a QFT computing device (e.g., a quantum optical computer), using the dynamic measure of *maximum entropy,* and the associated dynamic qubit, as an evaluation function of the happened convergence between the bottom-up/top-down concurrent computations. Let us deepen more accurately this essential point.

In this context, the key-role of the AFA axiom is threefold.

1. Before all, it grants the *compositionality* of the set inclusion relations by prohibiting that the ordinary transitivity rule (TR), $\langle \forall u,v,w \ \ ((uRv \wedge vRw) \rightarrow uRw)\rangle$, – where R is the inclusion relation and u, v, w are sets – holds in set inclusions, because TR supposes the set total ordering. In this way, because only the "weaker" transitivity of the Euclidean rule (ER) $\langle \forall u,v,w \ \ ((uRv \wedge uRw) \rightarrow vRw)\rangle$ between inclusions is here allowed, this means that the representation of sets ordered by inclusion as *oriented graphs,* in which the nodes are sets and the edges are inclusions with one only root (in our case the set u), satisfies *always* an "ascendant-descendant relationship" without "jumps" (each descendant has always its own ascendant, i.e., they form a *tree*). This is the core of the "compositionality" of the *inclusion operator* of a coalgebra defined on NWF sets, i.e., the basis of the so-called "tree-unfolding" of NWF sets, starting from an "ultimate root" similar to the *universal set V* – which is here allowed, because of the possibility of set self-inclusion[8] –, i.e., the disjunction of all sets forming the universe of the theory, like the "join" of a Boolean lattice. All this is the basis for extending the dual equivalence between the category **Stone** and the category **BAlg,** to the dual equivalence between the category of the coalgebras **Coalg** and the category of the algebras **Alg,** for an induced contravariant functor Ω^*, i.e., $\mathbf{Coalg(\Omega)} \leftrightharpoons \mathbf{Alg(\Omega^*)}$ [40][9].
2. Secondly, the AFA axiom and the "final coalgebra theorem" justify *the coalgebraic interpretation of modal logic* in the framework of *first-order logic* (see the fundamental Van Benthem's Theorem on this regard [41]) because the principle of set unfolding for partially ordered sets within an unbounded chain of set inclusions gives us an algebraically "natural" interpretation of the modal *possibility operator* "$\Diamond$", in the sense that $\langle\Diamond\alpha\rangle$ means that "α is true in *some* possible worlds" [42,43,44,45], so to give a computationally effective (first-order logic, where the predicate calculus is complete) justification to Thomason's early program of "reduction of the second-order logic to the modal logic" [46], made effective by another celebrated theorem, the Goldblatt-Thomason Theorem. Because any set tree can be modeled as a Kripke *frame*, this theorem defines rigorously which elementary classes of frames are modally definable (for a deep discussion of this theorem, see [47]. For an intuitive treatment of these notions, see sect. 5.2 in the Appendix). Because modal logic is the proper logic of deontic systems, this solves in principle A. Sen's problem of how implementing value systems in CCR's, both in economy and social computing (see §1.4)
3. Thirdly, in the fundamental paper of 1988 [48] Abramsky first suggested that the endofunctor of modal coalgebras is the so-called "Vietoris functor" $\mathcal{V}$ [10]. In this way we can extend the duality between coalgebras and algebras for the induction of a contravariant functor Ω^*, to the *dual equivalence* between modal coalgebras and modal algebras for the induction of a contravariant functor $\mathcal{V}^*$, i.e., $\mathbf{Coalg(\mathcal{V})} \leftrightharpoons \mathbf{Alg(\mathcal{V}^*)}$ [40]**.** This depends on the fact that $\mathcal{V}$ is a functor defined on a particular category of topological spaces, the category of the vector spaces **Vect.** Vector spaces are fundamental in physics: also the Hilbert spaces of the quantum physics mathematical formalism belong to such a category. The morphisms characterizing the vector space category are, indeed, linear functions, so if we apply to modal coalgebras Van Benthem's "correspondence theorem" [49] and the consequent "correspondence theory" [41] between the modal logic and the decidable fragments of the first order monadic predicate calculus, associating each axiom of modal calculus with a first order formula (see in Appendix 5.2 some exam-

on *finite* sets. It is evident that Zermelo's strategy of migrating to second-order set-theoretic semantics grants categoricity to mathematics on an *infinitistic* basis.

[8] Recall that set self-inclusion is not allowed for standard sets because of Cantor's theorem. This impossibility is the root of all semantic antinomies in standard set theory, from which the necessity of a second-order set-theoretic semantics ultimately derives.

[9] This depends on the trivial observation that a coalgebra $C = \langle C, \gamma : C \rightarrow \Omega C\rangle$, where γ is a transition function characterizing C, over an endofunctor Ω: $\mathbf{C} \rightarrow \mathbf{C}$ can be seen also as an algebra in the opposite category $\mathbf{C}^{\mathbf{op}}$, i.e., $\mathbf{Coalg(\Omega)} = (\mathbf{Alg(\Omega^{op})})^{\mathbf{op}}$ [40]

[10] The fundamental property of $\mathcal{V}$ is that it is the counterpart of the power set functor $\wp$ in the category of the topological spaces (i.e., for continuous functions) such as the Stone space category, **Stone**. This functor maps a set S to its power set $\wp(S)$ and a function $f : S \rightarrow S'$ to the image map $\wp f$ given by $(\wp f)\,(X) := f\,[X]$ $(= \{ f(x) \mid x \in X\})$. Applied to Kripke's relational semantics in modal logic, this means that Kripke's *frames* and *models* are nothing but "coalgebras in disguise". Indeed, a *frame* is a set of "possible worlds" (subsets, s) of a given "universe" (set, S) and a binary "accessibility" relation R between worlds, $R \subseteq S \times S$. A Kripke's *model* is thus a frame with an *evaluation function* defined on it. Now R can be represented by the function $R[\bullet]$: $S \rightarrow \wp(S)$, mapping a point s to the collection $R[s]$ of its successors. In this way frames in modal logic correspond to coalgebras over the *covariant* power set functor $\wp$. For such a reconstruction see [40].

ples), we obtain the following amazing result that Abramsky first suggested [48], and Kupke, Kurz & Venema developed [45].

4. Namely, we can formally justify the *modal coinduction (tree-unfolding) of predicate domains* – also of SWF and of SDF, indeed – of set *pre-orders* that with respect to Sen's CCR's arrives till the definition of *class equivalence* in social and economic rational choices, and not simply of "indifference between preferences" (see §1.4), also in *infinite data streams* (see §1.5). This is essentially due to the "Euclidean rule" $\langle \forall u,v,w \ \ ((uRv \wedge uRw) \rightarrow vRw) \rangle$, as substituting the "transitive rule" in NWF set inclusions. This allows us, in the coalgebraic simulation of dynamic social/economic systems, to define class equivalences, also for specific CCR's (see Appendix 5.1), on pre-orders among triples of sets (see def. 7 in §1.4 with the Euclidean rule substituting the transitive one). We can indeed justify *the modal operators* of the "possible converse membership" or "possible co-membership", $\langle \ni \rangle$, and of the "actual co-membership", i.e., $\neg\langle \neg \ni \rangle$, that is, $[\ni]$, where the angular and square parentheses are reminders of the possibility-necessity, "◊-□" operators, respectively [40].
5. For social computing, this means that we are in principle able to model *in real time* on the data flow, the formation/dissolution of coherent domains of CCR's, according to definite criteria of social/economic motivations, interests, values, beliefs. They correspond to the constitution/dissolution of new social collective actors, as far as the boundary conditions in the social/economic environment allow their *far from equilibrium* stability.

What, intuitively, all this means for our aims is that, because modal coalgebras admit only a *stratified (indexed) usage* of the necessity operator □ and of the universal quantifier ∀, a set *actually* exists as far as effectively *unfolded* by a co-inductive, concurrent procedure – not "constructed" by an inductive, algebraic one. For this reason, we are able to deal with infinite data streams, and finally to conceive *realistic* social and economic models!

In this framework, the semantic evaluations in the Boolean logic effectively consist in a *convergence* between an inductive "constructive" procedure, and a co-inductive "unfolding" procedure, also over unbounded chains of set inclusions.

Namely, they effectively consist in the superposition *limit/colimit* between two concurrent inductive/coinductive computations (see Appendix 5.1). This is the core of Abramsky notion of *finitary objects* as "limits of finite ones", definable only on NWF sets These finitary objects are indeed, according to him, the most proper objects of the mathematical modelling of realistic computations, also in social computing, we can add [48].

This is also the core of the related notion of *duality* between an *initial algebra*, starting from a *least* fixed-point, $x = f(x)$, and its *final coalgebra,* starting from a *greatest* fixed-point (see Appendix 5.1), at the basis of the notion of *Universal Coalgebra* as a "general theory of both computing and dynamic systems" [39]. This theory allows to justify a *formal semantics of computer programming* as satisfaction of a given program onto the states of a computing system, outside the Turing paradigm. Indeed, this approach systematically avoids the necessity of referring to an UTM for justifying formally the *universality* in computations, because of the possibility of referring to the algebraic and co-algebraic universality[11]. At the same time, this theory is able to give a *strong formal foundation* to the notion of *natural computation,* as far as we extend such a coalgebraic semantics to quantum systems and quantum computation. This research program has been inaugurated by S. Abramsky and his group at Oxford only few years ago, both in fundamental physics [50], and in QM computing [51], even though it has its most natural implementation in a QFT foundation of both quantum physics and quantum computation [52][12].

Finally, this gives a logical interpretation as a *predicate* (e.g., "being horse") of the "doubled number", i.e., $\mathcal{N}_A, \mathcal{N}_{\tilde{A}}$, as identity functions relative to two mirrored (doubled) sets of degrees of freedom, A and $\tilde{A}$, one relative to a logical realm (the Algebra(Ω*)), the other to its dual input realm (the Coalgebra(Ω)), the latter satisfying (making true) *naturally* – i.e., *dynamically* in this QFT implementation – the former (see above, sect. 2.3). The co-membership relation in the coalgebraic half has its physical justification in QFT by the general principle of the "*foliation* of the QV" at the ground state, and of the relative Hilbert space into physically inequivalent subspaces", allowing "the building up" via SSB of ever more complex phase coherence domains in the QV, given their stability in time. They do not depend, indeed, on any energetic input (they depend on as many NGB condensates $|\mathcal{N}|$, each correspondent to a SSB of the QV at the ground state), but on as many "entanglements" with stable structures of the environment [53]. This justifies Freeman and Vitiello in suggesting that this is the fundamental mechanism of the formation of the so called "long-term memory" traces in brain dynamics [54], i.e., the formation of the "deep beliefs" in our brains by which each of us interprets the world, based on her/his past experience, using the AI recent diffused jargon in the artificial neural network computing [55].

Anyway, apart from this "ontological" exemplification, useful however to connect the present discussion with the rest of this paper, all this means extending to a Boolean lattice L of the monadic predicate logic the modal semantics notions of co-induction and/or of "tree unfolding", so to give the formal justification of the modal notion of "local truth" also in a computational environment. Indeed, because such a co-inductive procedure of predicative domains justification is defined on NWF sets supporting set self-inclusion, i.e., $x \rightarrow \{x\}$, for each of these co-induced domains also the relative Id_x, i.e., the relative predicate φ is defined, without any necessity of referring to Fregean second-order axioms, such as the comprehension axiom of ZF set-theory, i.e.: $<\forall x \exists y \ x \in y \equiv \varphi x>$. This justifies the general statement that in CT coalgebraic semantics there exists a Tarski-like model theory [29], without, however, the necessity of referring to higher order languages for justifying the semantic meta-language [56], according to Thomason's reduction program.

[11] However, see the fundamental remarks about the limits of *decidability* and *computability* in this first-order modal logic semantic approach in [47], in which it is said, just in the conclusion, that one of the most promising research program in this field is related with the coalgebraic approach to modal logic semantics.

[12] This depends on the fact that *contravariance* in QM algebraic representation theory can have only an *indirect justification*, as Abramsky elegantly explained in his just quoted paper. QM algebraic formalism is, indeed, intrinsically based on Von Neumann's *covariant* algebra, so that only Hopf algebras' self-duality are "naturally" (in the algebraic sense of the allowed functorial transforms) justified.

Quoting the first concluding remark of V. Goranko and M. Otto contribution to the *Handbook of modal logic* devoted to model theory of modal logic [57], we can conclude too:

> Modal logic is local. Truth of a formula is evaluated at a current state (possible world); this localization is preserved (and carried) along the edges of the accessibility relations by the restricted, relativized quantification corresponding to the (indexed) modal operators.

3.3 Coalgebraic semantics of quantum systems

As a final step, let us apply to our QFT model of quantum computing system the coalgebraic model of computation over infinite data streams in terms of a particular abstract machine: *the infinite state black-box machine*.

In the light of the precedent discussion it is necessary and sufficient for such an aim to demonstrate that the collections of the "*q*-deformed Hopf algebras" and the "*q*-deformed Hopf coalgebras" of the QFT mathematical formalism constitute two *dually equivalent categories* for the contravariant application of the same functor *T,* that is, the contravariant application of the so-called *Bogoliubov transform* [9]. This is the classical QFT operator of "particle creation-annihilation", where the necessity of such a contravariance depends on the constraint of satisfying anyway the *energy balance principle.* I.e., ***q*-HAlg(*T*)** $\rightleftharpoons$ ***q*-HCoalg(*T**)**.

The complete justification of a coalgebraic interpretation of this mathematical formalism is given elsewhere [52], because we cannot develop it here. Nevertheless, at least two points of such a justification are important to emphasize, for justifying the interpretation of the maximal entropy in a QFT system as a semantic measure of information, i.e., as a statistical measure of *maximal local truth* in a CT coalgebraic logic for QFT systems.

Firstly, the necessary condition to be satisfied in order that a coalgebra category for some endofunctor Ω, i.e., **Coalg(Ω),** can be interpreted as a *dynamic* and/or *computational* system, is that it satisfies the formal notion of *state transition system* (STS). Generally, a STS is an abstract machine characterized as a pair $(S, \rightarrow)$, where *S* is a set of states, and $((\rightarrow) \subseteq S \times S)$ is a transition binary relation over *S.* If *p, q* belong to *S*, and (*p*, *q*) belongs to $(\rightarrow)$, then $(p \rightarrow q)$, i.e., there is a transition over *S.* For allowing that a dynamic/computational system be represented as a STS on a functorial coalgebra for some functor Ω it is necessary that the functor admits a *final coalgebra* [40]. I.e.:

Definition 1: (Definition of final coalgebra for a functor). A functor $\Omega : \mathbf{C} \rightarrow \mathbf{C}$ is said to admit a final coalgebra iff the category **Coalg(Ω)** has a final object, that is, a coalgebra $\mathbb{Z}$ such that from every coalgebra $\mathbb{A}$ in **Coalg(Ω),** there exists a unique homomorphism, $!\mathbb{A} : \mathbb{A} \rightarrow \mathbb{Z}$.

This property has a very intriguing realization – and this is the sufficient condition to satisfy for formalizing a QFT system as a computing system – into the final coalgebra associated with a particular abstract machine, the so-called "infinite state black-box machine" $\mathbb{M}\langle M,\mu\rangle$ [40]. It is characterized by the fact that the machine internal states, x_0, x_1, …, cannot be directly observed, but only some their values ("colors", c_n) associated with a state transition μ. I.e., $\mu(x_0) = (c_0, x_1)$, $\mu(x_1) = (c_1, x_2)$, … In this way, the only "observable" of this dynamics is the infinite sequence of behaviors or *stream* $beh\,(x_0) = (c_0, c_1, c_2, \ldots) \in C^{\omega}$ of value combinations or "words" over the data set *C*. The collection C^{ω} forms a *labeled* STS for the functor $C \times \mathcal{I}$, where $\mathcal{I}$ is the set of all the identity functions (labels), as far as we endow C^{ω} with a transition structure γ splitting a stream $u = c_0c_1c_2, \ldots$ into its "head" $h(u) = c_0$, and its *tail* $t(u) = c_1c_2c_3\ldots$. If we pose $\gamma(u) = (h(u),t(u))$, it is possible to demonstrate that the behavior map $x \mapsto beh\,(x)$ is the unique homomorphism from $\mathbb{M}$ to this coalgebra $\langle C,\gamma\rangle$, that is the final coalgebra $\mathbb{Z}$ in the category **Coalg(**$C \times \mathcal{I}$**)**[13]**.**

The abstract machine $\mathbb{M}$ is used in TCS for modelling the *coalgebraic semantics* of programming relatively to infinite data sets – the so-called data *streams*: think, for instance,at internet and more generally at all the ever-growing databases ("big data") [39]. The application of $\mathbb{M}$ for characterizing the QFT dynamics as a "computing dynamics" is evident in the light of the precedent discussion because we are allowed to interpret the thermodynamic functor *T* (Bogoliubov transform) characterizing the category ***q*-HCoalg(*T*)** as a functor able to associate the observable *c* of each "word" (phase coherence domain) of the QFT infinite dataset *C*, i.e., the infinite CCR's characterizing the QV, with the correspondent I_c, so that $T = (C \times \mathcal{I})$. Indeed, each I_c corresponds in the QFT formalism to the NGB condensate numerical value $|\mathcal{N}|$ identifying univocally each phase coherence domain, i.e. a "word" of the QV "language". In this way, the QV, because endowed with the SSB state-transition – effectively a phase-transition – structure γ, selecting every time one CCR (*head*) as to the rest of the others (*tile*), corresponds to the final coalgebra $\mathbb{Z}$ of the category ***q*-HCoalg(*T*)**. Moreover, the dynamics of the $\mathbb{M}_{QFT}$ is a *thermo*-dynamics*,* i.e., its state (phase) transition is "moved" by the II Principle (energy equipartition), in a way that must satisfy, on one hand, the "energy arrow contravariance" related to the I Principle, and, on the other one, without consuming all the QV energy "reservoir" as requested by the III Principle[14]. All this implies the necessity of doubling the behavior map, i.e., $x \mapsto beh\left(x,\tilde{x}\right)$, and all the related objects and structures – i.e., the necessity of "echoing" each word of the QV language –, so to satisfy finally the "dual equivalence" characterizing the QFT categorical formalism, i.e., ***q*-HAlg(*T*)** $\rightleftharpoons$ ***q*-HCoalg(*T**)**. In logical terms, the functor induction $T \leftarrow T^*$ means that the semantics (coalgebra) induces its own syntax (algebra). This, if justifies, on one hand, the computer scientist interest toward a coalgebraic approach to quantum computation for managing streams, on the other one, it demonstrates that the QFT interpretation of this approach is the more promising one. In fact, what we intended using the metaphor of the

[13] In parenthesis, in the machine $\mathbb{M}$ the general coalgebraic principle of the *observational* (or *behavioral*) *equivalence* among states holds in the following way. Indeed, for every two coalgebras (systems) $\mathbb{S}_1, \mathbb{S}_2 \in \mathbf{Coalg}\left(C \times \mathcal{I}\right), \left(!c_{\mathbb{S}_1} = !c_{\mathbb{S}_2}\right) \Rightarrow \left(!x_{\mathbb{S}_1} = !x_{\mathbb{S}_2}\right)$. All the scholars agree that this has an immediate meaning for quantum systems logic and mathematics, as a further justification for a coalgebraic interpretation of quantum systems.

[14] A condition elegantly satisfied in the QFT formalism by the *fractal structure* of the systems phase space and, therefore, by the *chaotic character* of the macroscopic trajectories (phase transitions) defined on it, generally, and specifically in the dissipative brain dynamics [60]

"word echoing" within the model of the $\mathbb{M}_{QFT}$ is effectively the DDF principle determining the *dynamic choice,* observer-independent, of the structure (syntax) of the "composed Hilbert space" of a QFT system as based on the *dual equivalence* (semantics) of one pair q-HAlg(T) $\rightleftharpoons$ q-HCoalg(T^*) representing the system.

All this is related, with the second, final, observation,

4 CONCLUSIONS

In this paper, we started from the analysis of the notion of "social welfare functions" as "collective choice functions" in social computing. We analysed the main computational problems of these notions, related mainly to the usage of a modelling of social and economic systems inspired at the statistical mechanics' study of systems stable at equilibrium, according to the principle of Gibbs' gas thermodynamics, from pioneering Samuelson's studies on.

We proposed, on the contrary, a modelling of social/economic systems inspired at condensed matter thermodynamics for far from equilibrium stability conditions, as fa as based on QFT fundamental physics, interpreted as a thermal field theory.

The common coalgebraic dynamic modelling developed independently in QFT and in TCS for dealing with otherwise non computable semantic problems, such as computations on (infinite) data streams and program security in functional programming, can be fruitfully applied also to the study of CCR's in dynamic social computing.

Particularly, we showed that we can formally justify the *modal coinduction (tree-unfolding) of predicate domains* – also of SWF and of SDF, indeed – of set *pre-orders* that with respect to Sen's CCR's arrives till the definition of *class equivalence* in social and economic rational choices, and not simply of "indifference between preferences" (see §1.4), also on *infinite data streams* (see §2.5). The limitation to finite sets is indeed the non-realistic limitation of social and economic computing in the actual complex and "liquid" scenario. A limitation that can be in principle avoided by the definition of the principle of *dual equivalence* between algebras and coalgebras, defined on NWF sets, allowing by hypotheses unbounded chains of set inclusions on which Abramsky's notion of "finitary computations" can be formally defined, freeing TCS from the false dichotomy between finitistic and non-finitistic computations.

This initial result opens the way to new promising scenarios in quantum natural, social and artificial computation to be explored in the next future.

5 APPENDIXES

5.1 Induction and coinduction as principles of set definition and proof for Boolean lattices

The collection of clopen subsets of a Stone space, as to which a Boolean algebra is isomorphic, according to the Stone theorem is effectively an ultrafilter U (or the maximal filter F) on the power-set, $\wp(S)$, of the set S. Namely, it is the *maximal partially ordered set* (maximal poset) within $\wp(S)$ ordered by inclusion, i.e., $(\wp(S), \subseteq)$, with the exclusion of the empty set. Any filter F is *dual* to an *ideal I*, simply obtained in set (order) theory by inverting all the relations in F, that is, $x \le y$ with $y \le x$, and by substituting *intersections* with *unions*. From this derives that each ultrafilter U is dual to a greatest ideal that, in Boolean algebra, is also a *prime* ideal, because of the so-called *prime ideal theorem*, effectively a corollary of the Stone theorem, demonstrated by himself. All this, applied to the Stone theorem, means that the collection of partially ordered clopen subsets of the Stone space to which a Boolean algebra is isomorphic, corresponds to a Boolean logic complete lattice L for a *monadic first order predicate logic.* From this, the definition of *induction* and *coinduction* as dual principles of set definition and proof is immediate, as soon as we recall that the fixed-point of a computation F is given by the equality $x = F(x)$ [36]:

Definition 2 (sets inductively/co-inductively defined by F). For a complete Boolean lattice L whose points are sets, and for an endofunction F, the sets

$$F_{ind} := \bigcap\{x \mid F(x) \le x\}$$
$$F_{coind} := \bigcup\{x \mid x \le F(x)\}$$

are, respectively, the sets *inductively* defined by a *recursive F,* and *co-inductively* defined by a *co-recursive F*. They correspond, respectively, to the *meet* of the pre-fixed point and the *join* of the post-fixed points in the lattice L, i.e., the least and greatest fixed-points, if F is monotone, as required from the definition of the category **Pos** (see above, sect. 3.1).

Definition 3 (induction and co-induction proof principles). In the hypothesis of **Definition 2**, we have:

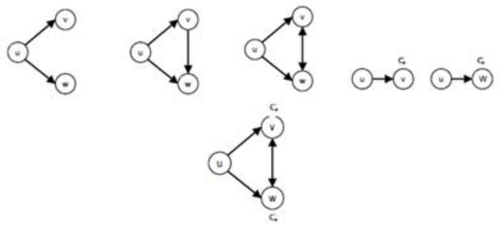
$$\textit{if } F(x) \le x \textit{ then } F_{ind} \le x \qquad \text{(induction as a method of proof)}$$
$$\textit{if } x \le F(x) \textit{ then } x \le F_{coind} \qquad \text{(co-induction as a method of proof)}$$

These two definitions are the basis for the *duality* between an *initial algebra* and its *final coalgebra,* as a new paradigm of computability, i.e., Abramsky's *finitary* one, and henceforth for the duality between the *Universal Algebra* and the *Universal Coalgebra* [39].

5.2 The extension of coinduction method to the definition of a complete Boolean Lattice of monadic predicates

The fundamental result of the above quoted Goldblatt-Thomason Theorem and Van Benthem Theorem is that a set-tree of NWF sets – effectively a set represented as an oriented graph where nodes are sets, and edges are inclusion relations with subsets governed by an Euclidean rule – corresponds to the structure of a Kripke *frame* of his relational semantics, characterized by a set of "worlds" and by a two-place accessibility relation R between worlds. E.g., the second graph from left below corresponds to the graph of the number 3, with $u = 3$, $v = 2$, $w = 1$. Therefore for understanding intuitively the extension of the coinduction method to the domains of monadic predicates of a Boolean lattice, let us start from 1) the "Euclidean rule (ER)" $\langle \forall u,v,w \ ((uRv \wedge uRw) \rightarrow vRw)\rangle$ (see the second from left graph below), driving all the NWF set inclusions and that is associated by Van Benthem's Correspondence Theorem to the modal axiom **E** (or **5**): $\langle \Diamond\alpha \rightarrow \Box\Diamond\alpha\rangle$, of the modal propositional calculus, and 2) from the "seriality rule (SR)" $\langle \forall u \exists v \ (uRv)\rangle$ (an example of this axiom is given by the fourth or the fifth graph below) – that has an immediate physical sense, because it corresponds to whichever energy conservation principle in physics, e.g., the I Principle of Thermodynamics –, and that is associated to the modal axiom **D:** $\langle \Box\alpha \rightarrow \Diamond\alpha\rangle$. The straightforward first order calculus, by which it is possible formally justifying the definition/justification by co-induction (tree unfolding) of an *equivalence class* as the domain of a given monadic predicate, through the application of the two above rules to whichever triple of objects $\langle u,v,w\rangle$, is the following.

For ER, $\langle\forall u,v,w\ ((uRv \wedge uRw) \rightarrow vRw)\rangle$; hence, for seriality, $\langle\forall u,v\ (u\mathrm{R}v \rightarrow v\mathrm{R}v)\rangle$; finally: $\langle\forall u,v,w\ [((uRv \wedge uRw) \rightarrow (vRw \wedge wRv \wedge vRv \wedge wRw)) \leftrightarrow ((v{\equiv}w) \subset u)]\rangle$. I.e., $(v{\equiv}w)$ constitutes an equivalence class, say **Y**, because a "generated" transitive[15]-symmetric-reflexive relation holds among its elements, which are therefore as many "descendants" of their common "ascendant", u. More intuitively, using Kripke's relational semantics graphs for modal logics, where $\langle u,v,w\rangle$ are as many "possible worlds" (models) of a given universe W, and where R is the two-place "accessibility relation" between worlds, the above calculus reads:



The final graph constitute a Kripke-like representation of the **KD45** modal system, also defined in literature as "secondary **S5**", since the equivalence relationship among all the possible worlds characterizing **S5** here holds only for a subset of them, that . In our example, the subset of worlds $\{w,v\}$.

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